

# IDENTIFYING NORMAL AND CONGRUENCE SUBGROUPS

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**ABSTRACT.** We study the arithmetic properties of the maps associated with the subgroups of finite indices of the Hecke group  $G_q$ . A necessary and sufficient condition for  $X \subseteq G_q$  to be normal is obtained. Such condition can be checked easily by **GAP**. As a byproduct, one may determine the normaliser of  $X$  in  $PSL(2, \mathbb{R})$  when  $q \neq 3, 4, 6$  and whether  $X$  is congruence when  $q \leq 6$ .

## 1. INTRODUCTION

1.1. Normal subgroups of the Hecke groups. Let  $q \geq 3$  be a fixed integer. The (inhomogeneous) Hecke group  $G_q$  is defined to be the maximal discrete subgroup of  $PSL(2, \mathbb{R})$  generated by  $S$  and  $T$ , where  $\lambda_q = 2\cos(\pi/q)$ ,

$$S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad T = \begin{pmatrix} 1 & \lambda_q \\ 0 & 1 \end{pmatrix}. \quad (1.1)$$

Normal subgroups  $X$  of finite indices of  $G_q$  have been studied extensively, and various necessary and sufficient conditions have been imposed on  $X$  for being normal (see [CS], [JS], [Ma], [Mc], [N] for examples). To the best of our knowledge, the verification of those conditions is quite delicate and can be lengthy. For instance, while it is well known that  $X$  is normal if and only if the action of  $G_q$  on  $G_q/X$  is fixed point free, the actual checking is very tedious as it involves some lengthy membership test.

1.2. The main results. The main purpose of this article is to associate with each subgroup  $X \subseteq G_q$  (given by generators and special polygons as in subsection 1.3) a finite subgroup  $G_X$  in the sense of Millington [Mi1, Mi2] and Atkin and Swinnerton-Dyer [AS] such that

- (i) the generators of  $G_X = \langle r_1, r_2 \rangle$  can be determined directly from the special polygon of  $X$  without matrix multiplication or any membership test and that the order of  $G_X$  can be calculated by **GAP**,
- (ii) whether  $X$  is normal in  $G_q$  can be determined by whether the order of  $G_X$  meets the index  $[G_q : X]$  (see Proposition 6.2).

As a byproduct, one may determine the normaliser of  $X$  in  $PSL(2, \mathbb{R})$  when  $q \neq 3, 4, 6$  (see subsection 7.2) and whether  $X$  is congruence when  $q \leq 6$  (see Section 8). Note that the normalisers of subgroups of finite indices of  $PSL(2, \mathbb{Z})$  in  $PSL(2, \mathbb{R})$  can be determined by studying the *big picture* of Conway (see [L]). The normalisers of subgroups of finite indices of  $G_4$  and  $G_6$  in  $PSL(2, \mathbb{R})$  are open to us. The permutations  $\sigma_0$  and  $\sigma_1$  of He and Read in their study of *dessins d'enfants* (pp.12 of [HR]) can be calculated easily by our results as  $\sigma_0 = r_1$  and  $\sigma_1 = r_2$ .

1.3. The main assumption. The main assumption of this article is that subgroups of finite indices of  $G_q$  are described as in Kulkarni [K]. Equivalently, in terms of fundamental domains and sets of independent generators in the sense of Rademacher. The existence of such description is guaranteed by Kurosh Theorem [Ku]. To be more precise, it is proved in [LLT1] that every subgroup  $X$  of finite index of  $G_q$  has a fundamental domain  $M_X$  which is a special polygon and that the set of side pairings of the Hecke-Farey symbols associated with the special polygon  $M_X$  is a set of independent generators of  $X$ .

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1.4. A simple idea. Jones and Singerman in their beautiful paper [JS] give a close relationship between subgroups of the *triangle groups* and *maps*. Following their insight, we make the special polygon  $M_X$  into a map and investigate the monodromy group  $G_X = \langle r_1, r_2 \rangle$  of  $M_X$ . As a homomorphic image of  $G_q$ ,  $G_X$  acts on the set of cosets  $G_q/G_0$ , where  $G_0$  is a map subgroup (see Definition 5.1). The key of our study is that  $M_X$  is constructed in such a way that (i)  $G_0$  is a conjugate of  $X$ , (ii) the permutations  $r_1$  and  $r_2$  are compatible with the action of  $S$  and  $R = ST^{-1}$  on  $G_q/X$  (see Lemmas 5.2-5.4), (iii)  $r_1$  and  $r_2$  can be written down without any membership test (see Examples 7.1-7.5). As a consequence, the conditions (i) and (ii) of subsection 1.2 are satisfied and we are able to determine the normality of  $X$  by checking whether  $|G_X|/[G_q : X]$  is one (Proposition 6.2).

1.5. The organisation. In Section 2, we give a lemma that will be used in Section 6. See [W] for more detail. The main theme of Section 3 is the geometrical construction of a special polygon (map) and a set of independent generators of  $X$ . Section 4 gives a very short study of the basic properties of maps that is useful for our study. Section 5 studies the action of  $G_X$  on  $G_q/X$ . Section 6 gives a simple test that enables us to identify normal subgroups of  $G_q$ . Section 7 demonstrates how normal subgroups can be identified and how the normaliser  $N(X)$  can be calculated. Example 7.5 gives a normal non-congruence subgroup of  $PSL(2, \mathbb{Z})$  of index 42 which is the smallest possible index of a normal non-congruence subgroup of  $PSL(2, \mathbb{Z})$ . Section 8 studies the congruence subgroup problem and Appendix A studies the torsion normal subgroups.

It is worthwhile to note that  $G_0(2) = \{(x_{ij}) \in G_6 : x_{12} \equiv 0 \pmod{2}\}$  has index 3 in  $G_6$  (see (ii) of Example 7.4). This is a little surprise to us as one expects, in line with the modular group case where a special polygon of  $\Gamma_0(2)$  is a 3-gon, that a special polygon of  $G_0(2)$  is a 6-gon and that  $[G_6 : G_0(2)] = 6$ .

## 2. A TECHNICAL LEMMA

Throughout the section,  $\Omega$  is a finite set and  $S_\Omega$  is the symmetric group on  $\Omega$ .

**Lemma 2.1.** *Suppose that  $G$  acts transitively on  $\Omega$ . Then  $C_{S_\Omega}(G) \cong N_G(G_d)/G_d$ , where  $G_d$  is the one point stabiliser of  $d \in \Omega$ .*

*Proof.* Since  $G$  is transitive, The action of  $G$  on  $\Omega$  is isomorphic to the action of  $G$  on the set of cosets  $G/G_d$ . Without loss of generality, we may assume that  $\Omega = G/G_d$  and that  $x(G_d) = xG_d$  for  $x \in G$ . Let  $x \in C_{S_\Omega}(G)$ . Then  $x(G_d) = e_x G_d \in G/G_d$  for some  $e_x \in G$ . For each  $g \in G_d$ , one has  $gx(G_d) = xg(G_d) = x(G_d)$ . Hence  $ge_x G_d = e_x G_d$ . This implies that  $e_x G_d \in N_G(G_d)/G_d$ . As a consequence, one can show that  $C_{S_\Omega}(G) \cong N_G(G_d)/G_d$  by studying the homomorphism  $\Phi : C_{S_\Omega}(G) \rightarrow N_G(G_d)/G_d$  defined by  $\Phi(x) = e_x^{-1} G_d$ . Note that for each  $r \in N_G(G_d)$ , the permutation defined by  $\sigma(gG_d) = gr^{-1}G_d$  commutes with  $G$  which implies that  $\Phi$  is surjective ( $e_\sigma = r^{-1}$ ,  $\Phi(\sigma) = rG_d$ ).  $\square$

**Corollary 2.2.** *Suppose that  $G$  acts transitively and freely (equivalently,  $G_x = 1$  for all  $x \in \Omega$ ) on  $\Omega$ . Then  $G \cong C_{S_\Omega}(G)$ .*

## 3. FUNDAMENTAL DOMAINS OF SUBGROUPS OF $G_q$

3.1. In [K], Kulkarni applied a combination of geometric and arithmetic methods to show that one can produce a set of independent generators in the sense of Rademacher for the congruence subgroups of the modular group, in fact for all subgroups of finite indices. His method can be generalised to all subgroups of finite indices of the Hecke groups  $G_q$ . In short, for each subgroup  $X$  of finite index of  $G_q$ , one can associate with  $X$  a set of Hecke-Farey symbols (HFS)  $\{-\infty, x_0, x_1, \dots, x_n, \infty\}$ , a special polygon (fundamental domain)  $M_X$ , and an additional structure on each consecutive pair of  $x_i$ 's of the four types described below :

$$x_i \underset{\circ}{\smile} x_{i+1}, \quad x_i \underset{\bullet}{\smile} x_{i+1}, \quad x_i \underset{a}{\smile} x_{i+1}, \quad x_i \underset{e_r}{\smile} x_{i+1}. \quad (3.1)$$

where  $a$  is a nature number and  $1 < r < q$  is a divisor of  $q$ . Each nature number  $a$  occurs exactly twice or not at all. Similar to the modular group, the actual values of the  $a$ 's is unimportant: it is the pairing induced on the consecutive pairs that matters.

- (0) A set of Hecke-Farey symbols (HFS) is a finite sequence of cyclically arranged numbers  $\{-\infty, x_0, x_1, \dots, x_n, \infty\}$  such that
  - (a)  $x_i \in \mathbb{Q}(\lambda_q)$ ,  $x_i = 0$  for some  $i$ ,  $0 \leq i \leq n$ ,
  - (b)  $x_i = a_i/b_i$  is in reduced form for every  $i$  (letting  $x_{-1} = -\infty = -1/0$  and  $x_{n+1} = \infty = 1/0$ ).
- (i) The side pairing  $\circ$  is the following elliptic element of order 2 that pairs the even line  $(a/b, c/d)$  with itself ( $cb - ad = 1$ ). The trace of such an element is 0. Recall that the  $G_q$ -translates of the hyperbolic line  $(0, \infty) = (0, i) \cup (i, \infty)$  are called the even lines.

$$\begin{pmatrix} c & a \\ d & b \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} c & a \\ d & b \end{pmatrix}^{-1} \in G_q. \quad (3.2)$$

- (ii) The side pairing  $\bullet$  is the following elliptic element of order  $q$  that pairs the odd line  $(a/b, c/d)$  with itself ( $cb - ad = 1$ ). The absolute value of the trace of such an element is  $\lambda_q$ . Recall that the  $G_q$ -translates of the hyperbolic line  $(0, e^{\pi i/q}) \cup (e^{\pi i/q}, \infty)$  are called the odd lines.

$$\begin{pmatrix} c & a \\ d & b \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & \lambda_q \end{pmatrix} \begin{pmatrix} c & a \\ d & b \end{pmatrix}^{-1} \in G_q. \quad (3.3)$$

- (iii) The two even lines  $(a/b, c/d)$  and  $(u/v, x/y)$  with the label  $a$  are paired together by the following element of infinite order ( $cb - ad = 1$ ,  $vx - yu = 1$ ).

$$\begin{pmatrix} u & -x \\ v & -y \end{pmatrix} \begin{pmatrix} c & a \\ d & b \end{pmatrix}^{-1} \in G_q. \quad (3.4)$$

- (iv) The side pairing  $e_r$  pairs  $x_i = u/v$  and  $x_{i+1} = x/y$  ( $vx - yu \neq 1$ ) as follows. There exists  $y_2 < y_3 < \dots < y_{q-q/r}$  such that  $x_i, y_2, \dots, y_{q-q/r}, x_{i+1}$  are consecutive entries of a  $q$ -gon  $P$ . Let  $a/b$  and  $c/d$  (in reduced forms) be the smallest and largest entries of  $P$ . Then  $x_i$  and  $x_{i+1}$  are paired by

$$\begin{pmatrix} c & a \\ d & b \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & \lambda_q \end{pmatrix}^{q/r} \begin{pmatrix} c & a \\ d & b \end{pmatrix}^{-1} \in G_q. \quad (3.5)$$

See subsection 3.3 and Example 7.4 for such side parings of order  $r$ .

- (v) The special polygon  $M_X$  associated with the HFS is a fundamental domain of  $X$ . It is a union of  $q$ -gons and  $s$ -gons (each  $\bullet$  gives a 1-gon and each  $e_r$  gives a  $q/r$ -gon, see Example 7.4). Each  $s$ -gon has  $s$  special triangles. The side pairings  $I = \{\sigma_1, \sigma_2, \dots, \sigma_m\}$  associated with the HFS is a set of independent generators of  $X$ . The number  $d$  of special triangles (a special triangle is a fundamental domain of  $G_q$ ) of the special polygon is the index of the subgroup  $X$ .
- (vi) The set of independent generators consists of  $s$  matrices of infinite order, where  $s$  is the number of the nature number  $a$ 's in the Hecke-Farey symbols.
- (vii) The subgroup  $X$  has  $\tau_2$  (the number of the circles  $\circ$  in HFS) inequivalent classes of elliptic elements of order 2 that are conjugates of  $S$ . Each class has exactly one representative in  $I$ .
- (viii) The subgroup  $X$  has  $v_q$  (the number of the bullets  $\bullet$  in HFS) inequivalent classes of elliptic elements of order  $q$  that are conjugates of  $ST^{-1}$ . Each class has exactly one representative in  $I$ .
- (ix) The subgroup  $X$  has  $v_r$  (the number  $e_r$ 's in HFS) inequivalent classes of elliptic elements of order  $r$  that are conjugates of  $(ST^{-1})^{q/r}$ . Each class has exactly one representative in  $I$ .
- (x) The Hecke-Farey symbols (cusps) can be partitioned into  $v_\infty$  classes under the action of the set of independent generators, such equivalence classes are called the *vertices* of  $M_X$ . It follows that  $M_X$  has  $v_\infty$  vertices. The equivalence classes of even lines

are called the *edges* of  $M_X$ . The  $s$ -gons (see (v)) are called the *faces* of  $M_X$ . The set of vertices, edges and faces are denoted by  $V$ ,  $E$  and  $F$  respectively.

- (xi) The degree of the vertex  $x$ , denoted by  $w(x)$ , is the number of even lines in  $M_X$  that comes into  $x$ . Algebraically, it is the smallest positive integer  $m$  such that  $\pm T_q^m$  is conjugate in  $G_q$  to an element of  $X$  fixing  $x$ . Denote by  $V$  the set of vertices of  $X$ . Then  $[G_q : X] = \sum_{x \in V} w(x)$ .
- (xii) The genus  $g$  of  $M_X$  is given by the following Riemann Hurwitz formula.

$$2g - 2 + \tau_2/2 + \sum v_r(1 - 1/r) + v_\infty = [G_q : X](1/2 - 1/q), \quad (3.6)$$

where the sum is taken over the set of all positive divisors of  $q$ .

- (xiii) Let  $P$  be an ideal  $q$ -gon with cusps  $\{x_1, x_2, \dots, x_q\}$  (if  $\infty$  is a cusp, then  $x_1 = -\infty = -1/0$  if  $P$  lies to the left of the  $y$ -axis and  $x_q = \infty = 1/0$  if  $P$  lies to the right of the  $y$ -axis). Let  $x_i = a_i/b_i$  be in reduced form. Then

$$\begin{pmatrix} a_1 & -a_q \\ b_1 & -b_q \end{pmatrix} A^i = \begin{pmatrix} a_{i+1} & a_i \\ b_{i+1} & b_i \end{pmatrix}, \quad \text{where } A = \begin{pmatrix} \lambda_q & 1 \\ -1 & 0 \end{pmatrix}. \quad (3.7)$$

In the case  $q$  is a prime or  $X$  is torsion free, the side pairing  $e_r$  does not exist and the entries of the Hecke-Farey symbols in (0) of the above satisfy  $a_{i+1}b_i - a_ib_{i+1} = 1$  for all  $i$ . See [LLT1] for more detail. The numbers in (vi)-(xii) are called the *geometric invariance* of  $X$ .

**3.2. Subgroups of  $PSL(2, \mathbb{Z})$  of index 7.** It is well known that  $PSL(2, \mathbb{Z})$  has altogether 42 subgroups of index 7 (see [R], [LLT2]). Since  $PSL(2, \mathbb{Z}_7)$  has 14 subgroups of index 7, fourteen of the 42 subgroups of index 7 are congruence of level 7. The remaining 28 are non-congruence. A special polygon of such a subgroup  $A$  consists of two 3-gons and one special triangle. Since a special polygon of  $A$  has a special triangle,  $A$  has an element  $g$  of order 3. Suppose that  $A$  is normal. It follows that  $A$  contains all the conjugates of  $g$ . In particular,  $ST^{-1}, T^{-1}S \in A$ . Hence a special polygon of  $A$  is a union of two special triangles  $\{-\infty \underset{\bullet}{\cup} 0 \underset{\bullet}{\cup} \infty\}$ . This contradicts the fact that  $A$  has index 7. Hence  $A$  is not normal. As a consequence,  $A$  is self-normalised and has 7 conjugates. We shall list for each conjugacy class a representative as follows. Note that the subscripts refer to the degrees of the vertices.

$$M_7 = \{-\infty \underset{\bullet}{\cup} -1 \underset{\circ}{\cup} 0 \underset{\circ}{\cup} 1 \underset{\circ}{\cup} \infty\}. \quad (3.8)$$

$$M'_7 = \{-\infty \underset{\circ}{\cup} -1 \underset{\bullet}{\cup} 0 \underset{\circ}{\cup} 1 \underset{\circ}{\cup} \infty\}. \quad (3.9)$$

Subgroups (3.8) and (3.9) are congruence subgroups of level 7. The following subgroups are non-congruence.

$$M_{1,6} = \{-\infty \underset{\bullet}{\cup} -1 \underset{\circ}{\cup} 0 \underset{1}{\cup} 1 \underset{1}{\cup} \infty\}. \quad (3.10)$$

$$M'_{1,6} = \{-\infty \underset{\circ}{\cup} -1 \underset{\bullet}{\cup} 0 \underset{1}{\cup} 1 \underset{1}{\cup} \infty\}. \quad (3.11)$$

$$M_{3,4} = \{-\infty \underset{\bullet}{\cup} -1 \underset{1}{\cup} 0 \underset{\circ}{\cup} 1 \underset{1}{\cup} \infty\}. \quad (3.12)$$

$$M_{2,5} = \{-\infty \underset{\bullet}{\cup} -1 \underset{1}{\cup} 0 \underset{1}{\cup} 1 \underset{\circ}{\cup} \infty\}. \quad (3.13)$$

The geometric invariance of the above subgroups can be determined by (vi)-(xii) of subsection 3.1. For instance, the invariance of the subgroup (3.8) are  $\tau_2 = 3$ ,  $v_3 = 1$ ,  $v_\infty = 1$ , the degree of the only vertex is 7. The genus of the surface is zero.  $M_{1,6}$  will be used to construct a normal non-congruence subgroup of index 42 (see Example 7.5).

**3.3. A subgroup of the Hecke group  $G_6$ .** Let  $\lambda = \sqrt{3}$  and let  $M_X$  be given as follows.

$$M_X = \{-\infty \underset{\circ}{\cup} 0/1 \underset{\circ}{\cup} 1/\lambda \underset{\circ}{\cup} \lambda/2 \underset{\circ}{\cup} \infty\}. \quad (3.14)$$

An easy calculation (see (xiii) of subsection 3.1) shows that  $\{\lambda/2, 2/\lambda, \lambda/1, \infty\}$  are consecutive entries of the 6-gon  $P = \{-\infty, 0/1, 1/\lambda, \lambda/2, 2/\lambda, \lambda/1, \infty\}$ . By (i) and (iv) of subsection 3.1, a set of independent generators is given by

$$X = \left\langle S, \begin{pmatrix} 1 & 0 \\ \lambda & 1 \end{pmatrix} S \begin{pmatrix} 1 & 0 \\ \lambda & 1 \end{pmatrix}^{-1}, \begin{pmatrix} \lambda & 1 \\ 2 & \lambda \end{pmatrix} S \begin{pmatrix} \lambda & 1 \\ 2 & \lambda \end{pmatrix}^{-1}, \begin{pmatrix} 0 & 1 \\ -1 & \lambda \end{pmatrix}^3 \right\rangle. \quad (3.15)$$

Note that  $[G_6 : X] = 3$ . See Example 7.4 (Figure 3) for a picture of  $M_X$ .

#### 4. MAPS OF SUBGROUPS OF $G_q$

**4.1.  $M_X$  is a map.** Let  $M_X$  be a special polygon of  $X$  given as in Section 3.  $M_X$  is an orientation-preserving compact Riemann surface. Recall that (see (x) of subsection 3.1)

- (i)  $V$  = set of vertices of  $M_X$  = the set of equivalence classes of cusps of  $M_X$ ,
- (ii)  $E$  = set of edges of  $M_X$  = the set of equivalence classes of even lines of  $M_X$ ,
- (iii)  $F$  = set of faces of  $M_X$  = the set of  $r$ -gons of  $M_X$ .

An even line  $(x, y) = (a/b, c/d)$  paired by an elliptic element of order 2 (see (i) of subsection 3.1) gives an edge with one vertex only. Such an edge is called a *free edge*. A non-free edge  $(x, y)$  with only one vertex is called a *loop* (the end points  $x$  and  $y$  are paired by a set of side pairings of orders  $\geq 3$ ). An edge with two vertices is called a *segment*.

It is clear that the faces are simply connected. As a consequence,  $M_X = (V, E, F)$  is a *map* (a map on a compact orientable surface  $\mathcal{S}$  is an embedding of a finite connected graph  $\Gamma$  on  $\mathcal{S}$  such that the connected components of  $\mathcal{S} \setminus \Gamma$  are simply connected, see [CS] for more detail).  $\Gamma_X = (V, E)$  is called the graph of  $M_X$ .

**4.2. Directed graph and darts of  $M_X$ .** Let  $e$  be an edge of the map  $M_X$ . We may associate with the edge  $e$  two directed edges if  $e$  is not a free edge (see subsection 4.1) and one directed edge if  $e$  is a free edge as follow.

- (i) Suppose that  $e$  is a loop. Then one of the directed edge travels clockwise and the other counter-clockwise. In the case  $e$  is a segment with vertices  $x$  and  $y$ , one travels from  $x$  to  $y$  and the other travels from  $y$  to  $x$ .
- (ii) Suppose that  $e$  is a free edge. Then we associate with  $e$  one directed edge only. Such a directed edge always points to the vertex of the edge.
- (iii) A directed edge is called a *dart*. Denote by  $\Omega$  the set of all darts.

**4.3. The cosets  $G_q/X$  and the darts  $\Omega$  have the same cardinality.** Let  $\Phi$  be the special triangle with vertices  $\infty$ ,  $0$  and  $e^{\pi i/q}$ . Then  $\Phi$  is a fundamental domain of  $G_q$ . Suppose that  $G_q = \cup_{k=1}^r Xg_k$ . Replace  $g_k$  by  $x_k g_k$  for some  $x_k \in X$  if necessary, we may assume that  $\cup_{k=1}^r g_k \Phi = M_X$  (as a special polygon for  $X$ ). Each dart  $\hat{d}_k \in \Omega$  belongs to a unique special triangle  $g_k \Phi \in M_X$  (following the orientation). As a consequence, there exists a one to one correspondence  $\sigma$  between  $\Omega$  and  $G_q/X$  defined by  $\sigma(\hat{d}_k) = Xg_k$ . Note that each non-free edge belongs to exactly two special triangles.

#### 5. MONODROMY GROUPS OF SUBGROUPS OF $G_q$

**5.1. Monodromy groups of  $M_X$ .** Let  $\Omega$  be the set of darts of  $M_X$ . Define  $r_1$  to be the permutation that transposes the two darts of each loop and segment, and fixes the single dart of each free edge. Around each vertex  $v$  of  $M_X$ , the orientation of  $M_X$  imposes a cyclic ordering of the darts pointing towards  $v$  and  $r_0$  is the permutation with these as their

disjoint cycles. Define  $r_2 = r_1^{-1}r_0^{-1}$ .  $r_2$  is a permutation whose cycles correspond to the faces of  $M_X$ , again following the orientation. Define

$$G_X = \langle r_1, r_2 \rangle = \langle r_0, r_1 \rangle. \quad (5.1)$$

The permutations  $r_1$  and  $r_2$  can be written down easily from  $M_X$ . See Examples 7.1 and 7.2 for the actual readings of  $r_1$  and  $r_2$ . Note that the incidence relations of  $(V, E, F)$  are completely determined by  $r_0$  and  $r_1$  and that every dart appears in exactly one cycle of the  $r_i$ 's ( $0 \leq i \leq 2$ ). Let  $R = ST^{-1}$ . Since  $\{S, R\}$  is a set of independent generators of  $G_q$ ,  $o(R) = q$ , and the order of  $r_2$  is a divisor of  $q$ , the map  $f$  defined by

$$f(S) = r_1, f(R) = r_2 \quad (5.2)$$

is a surjective homomorphism from  $G_q$  to  $G_X$ . It is clear from (5.2) that  $G_q$  acts on  $\Omega$  and that the action of  $G_X$  on  $\Omega$  is isomorphic to the action of  $G_q$  on  $G_q/G_0$ , where  $G_0$  is any one point stabiliser of the action of  $G_q$  on  $\Omega$ .

**Definition 5.1.** The group  $G_X$  defined in (5.1) is called the *monodromy group* of  $X$ . The one point stabilisers of the action of  $G_q$  on  $\Omega$  are called the *map subgroups* of  $M_X$ . Denote by  $Y$  the kernel of  $f$ .  $Y$  is called the *normal map subgroup* associated with  $M_X$ .

**5.2. Map subgroups of  $X$ .** Since the map  $M_X$  is a special polygon of  $X$ , it is very natural to expect that the map subgroups of  $M_X$  are the conjugates of  $X$  (Lemma 5.4) and that the action of  $S$  and  $R$  on  $G_q/X$  can be described by the action of  $G_X$  on  $\Omega$  (Lemmas 5.2 and 5.3).

**Lemma 5.2.** Let  $\hat{d}_k, g_k$  and  $r_1$  be given as in subsections 4.3 and 5.1. Suppose that  $r_1\hat{d}_i = f(S)\hat{d}_i = \hat{d}_j$ . Then  $Xg_iS = Xg_j$ .

*Proof.* Following the definition of  $f(S)$ ,  $\hat{d}_i$  and  $\hat{d}_j$  are the directed edges associated with the same edge  $e$  and that  $\hat{d}_i \in g_i\Phi$ ,  $\hat{d}_j \in g_j\Phi$  (see subsection 4.3 for notation).

Since  $\hat{d}_i \in g_i\Phi$ , one has  $e \in g_i\Phi$ . Hence  $g_i^{-1}e \in \Phi \cap S\Phi$ . Equivalently,  $e \in g_i\Phi \cap g_iS\Phi$ . This implies that  $\hat{d}_i \in g_i\Phi$  and that  $\hat{d}_j \in g_iS\Phi$ . Since each dart belongs to a unique special triangle of  $M_X$  (see subsection 4.3), one has  $x_i g_i S\Phi = g_j\Phi$  for some  $x_i \in X$ . Since  $G_q$  acts freely on the  $G_q$ -translates of  $\Phi$ , one has  $x_i g_i S = g_j$ . Hence  $Xg_iS = Xg_j$ .  $\square$

**Lemma 5.3.** Let  $\hat{d}_k, g_k$  and  $r_2$  be given as in subsections 4.3 and 5.1. Suppose that  $r_2\hat{d}_i = f(R)\hat{d}_i = \hat{d}_j$ . Then  $Xg_iR = Xg_j$ .

*Proof.* Note first that  $\hat{d}_i \in g_i\Phi$ ,  $\hat{d}_j \in g_j\Phi$  and that  $\hat{d}_i$  and  $\hat{d}_j$  (following the orientation) are consecutive terms of an  $s$ -gon of  $M_X$ . Since  $\hat{d}_i \in g_i\Phi$ , one has  $g_i^{-1}\hat{d}_i \in \Phi$ . Hence  $g_i^{-1}\hat{d}_i = \hat{e}$ , where  $e$  is the even line  $(0, \infty) \in \Phi$  and  $\hat{e}$  is a directed edge associated with  $e$ . It follows that  $g_i^{-1}\hat{d}_i = \hat{e}$  is a term of an  $s$ -gon  $F_s$  of  $g_i^{-1}Xg_i$ . The term next to  $g_i^{-1}\hat{d}_i = \hat{e}$  (following the orientation) in  $F_s$  is  $R\hat{e}$ . This implies that  $g_iR\hat{e} = \hat{d}_j \in g_j\Phi$ . As a consequence, one has

$$\hat{d}_j = g_iR\hat{e} = g_iRg_i^{-1}g_i\hat{e} = g_iRg_i^{-1}\hat{d}_i \in g_iRg_i^{-1}g_i\Phi = g_iR\Phi. \quad (5.3)$$

Since each dart belongs to a unique special triangle of  $M_X$  (see subsection 4.3), one has  $x_i g_i R\Phi = g_j\Phi$  for some  $x_i \in X$ . Hence  $Xg_iR = Xg_j$ .  $\square$

**Lemma 5.4.** The map subgroups of  $M_X$  are conjugates of  $X$ . The action of  $G_X$  on  $\Omega$  is isomorphic to the action of  $G_q$  on  $G_q/X$ . Further, the normal map subgroup  $Y = \cap gXg^{-1}$  is a subgroup of  $X$ .

*Proof.*  $G_q$  acts on  $\Omega$  by (5.2). Let  $G_0$  be the stabiliser of  $\hat{d}_i \in \Omega$ . It follows easily that the action of  $G_q$  on  $G_q/G_0$  is isomorphic to the action of  $G_X$  on  $\Omega$ . In particular, one has  $|G_q/G_0| = |\Omega|$ .

Suppose that  $f(g)\hat{d}_i = \hat{d}_i$ . By Lemmas 5.2 and 5.3, one has  $Xg_i g = Xg_i$ . Equivalently,  $g \in g_i^{-1}Xg_i$ . This implies that  $G_0$  (the stabiliser of  $\hat{d}_i$ ) is a subgroup of  $g_i^{-1}Xg_i$ . Since  $G_0 \subseteq g_i^{-1}Xg_i$ ,  $|G_q/G_0| = |\Omega|$  and  $|G_q/g_i^{-1}Xg_i| = |G_q/X| = |\Omega|$  (see subsection 4.3), one

concludes that  $G_0 = g_i^{-1} X g_i$ . As a consequence, the map subgroups (see Definition 5.1) are conjugates of  $X$  and the action of  $G_X$  on  $\Omega$  is isomorphic to the action of  $G_q$  on  $G_q/X$ . In particular,  $Y = \cap g X g^{-1} \subseteq X$ .  $\square$

## 6. NORMAL SUBGROUPS OF $G_q$

**6.1. The main results.** Let  $X$  be a subgroup of finite index of  $G_q$ . The main purpose of this subsection is to show that  $X$  is a normal subgroup of  $G_q$  if and only if  $|G_X| = [G_q : X]$ .

**Lemma 6.1.** *Let  $X$  be a subgroup of finite index of  $G_q$ . Then  $C_{S_\Omega}(G_X) \cong N_{G_q}(X)/X$ ,  $\text{Aut } M_X = C_{S_\Omega}(G_X)$ .*

*Proof.* Since the action of  $G_X$  on  $\Omega$  is isomorphic to the action of  $G_q$  on  $G_q/X$  and  $G_q$  acts transitively on  $G_q/X$ ,  $G_X$  acts transitively on  $\Omega$ . By Lemma 2.1,  $C_{S_\Omega}(G_X) \cong N_{G_X}(G_{\hat{d}})/G_{\hat{d}}$  where  $G_{\hat{d}}$  is the one point stabiliser of  $\hat{d} \in \Omega$ . Apply the fact that the action of  $G_X$  on  $\Omega$  is isomorphic to the action of  $G_q$  on  $G_q/X$  one more time, one has  $N_{G_X}(G_{\hat{d}})/G_{\hat{d}} \cong N_{G_q}(X)/X$ .

$\text{Aut } M_X = C_{S_\Omega}(G_X)$  follows from the fact that the incidence relations of  $(V, E, F)$  is completely determined by  $G_X$  and that  $\sigma \in S_\Omega$  is an automorphism if and only if  $\sigma$  preserves the incidence relations of  $(V, E, F)$ .  $\square$

**Proposition 6.2.** *Let  $X$  be a subgroup of  $G_q$  of finite index and let  $G_X = \langle r_1, r_2 \rangle$ . Then  $X$  is a normal subgroup of  $G_q$  if and only if  $|G_X| = [G_q : X]$ . In the case  $X$  is normal,  $G_X \cong C_{S_\Omega}(G_X) \cong \text{Aut } M_X$ .*

*Proof.* Suppose that  $|G_X| = [G_q : X]$ . By (5.2), one has

$$|G_q/Y| = |G_q/\ker f| = |G_X| = [G_q : X]. \quad (6.1)$$

By Lemma 5.4,  $Y \subseteq X$ . Hence  $X = Y$  is normal. Conversely, suppose that  $X$  is a normal subgroup. By (5.2) and Lemma 5.4, one has  $G_X \cong G_q/Y = G_q/X$ . Further, one has  $G_X \cong C_{S_\Omega}(G_X)$  (see Corollary 2.2). This completes the proof of the proposition.  $\square$

**6.2. The geometric invariance of  $X$  and  $M_X$ .** The invariance of  $X$  and  $M_X$  (see (xiii) of subsection 3.1) can be described by the group  $G_X = \langle r_0, r_1, r_2 \rangle = \langle r_1, r_2 \rangle$  as follows.

- (i)  $|E|$  = the number of disjoint cycles (counting 1-cycles) in  $r_1$ ,  $[G_q : X] = |\Omega|$  = the number of entries (counting 1-cycles) in  $r_1$  and  $r_2$  = the number of 1-cycles of  $r_1$ .
- (ii)  $|F|$  = the number of disjoint cycles (counting 1-cycles) in  $r_2$ . For each divisor  $r$  of  $q$  ( $1 < r \leq q$ ),  $v_r$  = the number of  $q/r$  cycles in  $r_2$ .
- (iii)  $|V|$  = the number of disjoint cycles (counting 1-cycles) in  $r_0 = r_2^{-1} r_1^{-1}$ . Each cycle in  $r_0$  represents a vertex, the degree of the vertex is the length of the cycle.
- (iv)  $N_{G_q}(X)/X \cong C_{S_\Omega}(G_X) = \text{Aut } M_X$ .  $X$  is normal iff  $C_{S_\Omega}(G_X) \cong G_X \cong G_q/X$ .

## 7. FIRST APPLICATION : IDENTIFYING NORMAL SUBGROUPS

**7.1. Identifying normal subgroups.** Let  $X$  be a subgroup of  $G_q$  of finite index and let  $G_X$  be given as in (5.1). The construction of  $r_0$  is difficult as it is not an easy matter to list the directed edges of  $c_v$  according to the orientation of  $M_X$ . *However, it is an easy matter to write down the permutation representations of  $r_1$  and  $r_2$*  (see Examples 7.1-7.5). As a consequence, the group  $G_X$  can be determined by  $M_X$  without any membership test. The order of  $G_X = \langle r_1, r_2 \rangle$  can be determined by **GAP**. Hence whether  $X$  is normal can be determined by Proposition 6.2 and **GAP** [G].

**7.2. Normalisers and automorphisms.** Let  $N(X)$  be the normaliser of  $X$  in  $PSL(2, \mathbb{R})$ . In the case  $q \neq 3, 4, 6$ , the Margulis characterisation of arithmeticity in terms of the commensurator implies that  $N_{G_q}(X) = N(X)$ . Hence  $N(X)/X$  can be determined by Lemma 6.1. The automorphism group of the map  $M_X$  can be determined by Lemma 6.1 also.

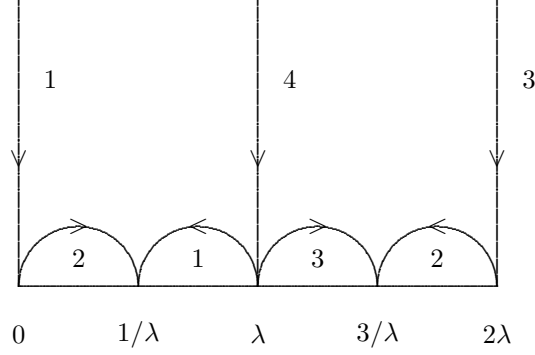


Figure 1

**7.3. Discussion.** (i) A map is called *quasi-regular* if every vertex has the same degree, every face has the same number of edges, and either it has no free edges or all its edges are free. It is clear that  $X$  is normal only if  $M_X$  is quasi-regular. The converse is not true (see Example 7.1). (ii) A map  $M$  is *regular* if  $\text{Aut } M$  acts transitively on the set of darts. A quasi-regular map on a genus zero surface is regular (see Corollary 6.4 of [JS]). (iii) By our result in subsection 4.3,  $|\Omega| = [G_q : X]$ . As a consequence,  $M_X$  is regular if and only if  $X$  is normal (see Lemma 6.1).

**Example 7.1.** Let  $\lambda = \sqrt{2}$  and let

$$M_X = \{-\infty \underset{1}{\smile} 0 \underset{2}{\smile} 1/\lambda \underset{1}{\smile} \lambda \underset{3}{\smile} 3/\lambda \underset{2}{\smile} 2\lambda \underset{3}{\smile} \infty\} \quad (7.1)$$

be the Hecke-Farey symbols of  $X \subseteq G_4$  of index 8 (see Figure 1). Applying (i)-(iii) of subsection 3.1, a set of independent generators is given by

$$g_1 = \begin{pmatrix} \lambda & 1 \\ 1 & \lambda \end{pmatrix}, \quad g_2 = \begin{pmatrix} 7 & -2\lambda \\ 2\lambda & -1 \end{pmatrix}, \quad g_3 = \begin{pmatrix} 3\lambda & -7 \\ 1 & -\lambda \end{pmatrix}, \quad (7.2)$$

where the side pairing  $g_r$  pairs the even lines with label  $r$ . One sees easily from Figure 1 that

$$r_2 = (1, 2, \bar{1}, \bar{4})(4, 3, \bar{2}, \bar{3}), \quad r_1 = (1, \bar{1})(2, \bar{2})(3, \bar{3})(4, \bar{4}), \quad (7.3)$$

where  $x$  and  $\bar{x}$  are the directed edges associated with the same edge. It follows that  $r_0 = r_2^{-1}r_1^{-1} = (2, 3, \bar{2}, 1)(\bar{4}, \bar{3}, 4, \bar{1})$ . By **GAP**, one has  $|G_X| = 16 > 8 = [G_4 : X]$ . By Proposition 6.2,  $X$  is not normal. Note that  $M_X$  is quasi-regular but not regular. Further,  $\text{Aut } M_X = C_{S_8}(G_X) \cong Z_2 \times Z_2$ . The following map is also quasi-regular.  $M_Y = \{-\infty \underset{1}{\smile} 0 \underset{3}{\smile} 1/2 \underset{1}{\smile} 2 \underset{2}{\smile} 3 \underset{3}{\smile} \infty\}$ . Note that  $[PSL(2, \mathbb{Z}) : Y] = 12$  and that  $Y$  is not normal.

**Example 7.2.** Let

$$M_X = \{-\infty \underset{1}{\smile} 0 \underset{2}{\smile} 1 \underset{2}{\smile} 3/2 \underset{3}{\smile} 2 \underset{3}{\smile} 3 \underset{1}{\smile} \infty\} \quad (7.4)$$

be the Hecke-Farey symbols of  $X \subseteq \Gamma = PSL(2, \mathbb{Z})$  of index 12 (see Figure 2). Applying (i)-(iii) of subsection 3.1, a set of independent generators is given by

$$g_1 = \begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix}, \quad g_2 = \begin{pmatrix} 2 & -3 \\ 3 & -4 \end{pmatrix}, \quad g_3 = \begin{pmatrix} 5 & -12 \\ 3 & -7 \end{pmatrix}. \quad (7.5)$$

One sees easily from Figure 2 that

$$r_2 = (1, 2, \bar{4})(4, 6, \bar{5})(5, \bar{3}, \bar{1})(\bar{6}, \bar{2}, 3), \quad r_1 = (1, \bar{1})(2, \bar{2})(3, \bar{3})(4, \bar{4})(5, \bar{5})(6, \bar{6}), \quad (7.6)$$





**Example 7.3.** Let  $\lambda = 2\cos\pi/5 = (1 + \sqrt{5})/2$  and let

be the Hecke-Farey symbols of  $X \subseteq G_5$  of index 10. Similar to Examples 7.1 and 7.2 (with appropriate labeling of the even lines), we have  $r_1 = (1, \bar{1})(2, \bar{2})(3, \bar{3})(4, \bar{4})(5, \bar{5})$  and  $r_2 = (1, 2, 3, 4, 5)(\bar{1})(\bar{2})(\bar{3})(\bar{4})(\bar{5})$ . In particular,  $M_X$  consists of one 5-gon and five 1-gon. By subsection 7.3,  $X$  is not normal. By **GAP**,  $C_{S_{10}}(\langle r_1, r_2 \rangle) = \langle (1, 2, 3, 4, 5)(\bar{1}, \bar{2}, \bar{3}, \bar{4}, \bar{5}) \rangle$ . Hence  $\text{Aut } M_X = C_{S_{10}}(\langle r_1, r_2 \rangle) \cong N(X)/X \cong Z_5$  (see subsection 7.2 and Lemma 6.1).

**Example 7.5. Normal non-congruence subgroups of index 42 of  $PSL(2, \mathbb{Z})$ .** Let  $X$  be the group associated with  $M_{1,6}$  and let  $Y$  be the normal map group of  $X$  (see (3.10) and Definition 5.1). By **GAP**, the group  $G_X$  has order 42. Hence  $Y$  is a normal subgroup of index 42. Since  $X$  is non-congruence,  $Y$  is also non-congruence. Note that 42 is the smallest possible index of a normal non-congruence subgroup of  $PSL(2, \mathbb{Z})$  (see pp. 276 of [N]).

The *principal congruence subgroup* of  $G_q$  associated with  $(r) \subseteq \mathbb{Z}[\lambda_q]$  is the subgroup  $G(r) = \{x \in G_q : x \equiv \pm I \pmod{r}\}$ .  $X \subseteq G_q$  is a *congruence subgroup* if  $G(r) \subseteq X$  for some  $r$ . Let  $X$  be a subgroup of finite index of  $G_q$  and let  $n$  be the least common multiple of the degrees of the vertices. The number  $n$  is called the *level* of  $X$ . It is known that if  $X \subseteq G_q$  has level  $n$ , then

- (i) if  $q = 3, 4, 6$ , then  $X$  is congruence if and only if  $G(n\lambda_q) \subseteq X$  (see [Wo], [P]),  
(ii) if  $q = 5$ , then  $X$  is congruence if and only if  $G(2n) \subseteq X$  (see [LL]).

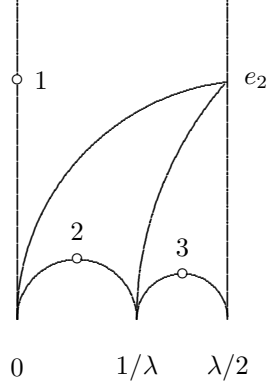


Figure 3

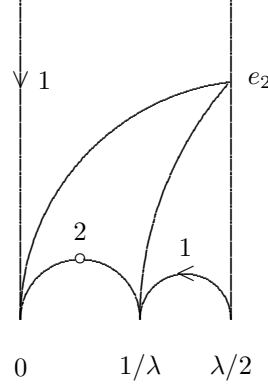


Figure 4

As a consequence, the algorithm in [LLT3] that determines whether a group is congruence extends essentially verbatim to the Hecke groups  $G_q$  when  $q \leq 6$  (see pp. 46 of [HR]). The weakness of their algorithm is that the calculation becomes very lengthy when the level  $n$  is large. In the case  $q = 3$ , to the best of our knowledge, the congruence test for  $G_3 = \Gamma = PSL(2, \mathbb{Z})$  developed by Tim Hsu [H] is the most effective one in the literature. His test can be implemented easily as long as a  $TU$ -representation of  $X$  is determined, where

$$T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, U = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}. \quad (8.1)$$

Recall that a  $TU$ -representation is a set of two permutations  $\{f(T), f(U)\}$ , where  $f(x)$  denotes the permutation representation of  $x \in \{T, U\}$  on  $PSL(2, \mathbb{Z})/X$ . The determination of  $f(T)$  and  $f(U)$  involves matrix multiplication and membership test. As this procedure may be tedious when the index  $[PSL(2, \mathbb{Z}) : X]$  is large (see Section 5.4 of [KL] for example), we suggest the following, which makes the determination of whether  $X$  is congruence very easy as long as  $X$  is given as in subsection 3.1.

**Proposition 8.1.** *Let  $G_X = \langle r_0, r_1, r_2 \rangle$  be the monodromy group of  $X \subseteq PSL(2, \mathbb{Z})$ . Then*

$$f(T) = r_0 = r_2^{-1} r_1^{-1}, \quad f(U) = r_1 r_0^{-1} r_1^{-1}. \quad (8.2)$$

*$X$  is a congruence subgroup if and only if  $f(T)$  and  $f(U)$  satisfy the conditions given in Section 3 of [H].*

*Proof.* By (5.2) and Lemma 5.4,  $r_1$  gives the action of  $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$  and  $r_2$  gives the action of  $\begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}$  on  $PSL(2, \mathbb{Z})/X$ . It follows by direct calculation that  $f(T) = r_0$  and that  $f(U) = r_1 r_0^{-1} r_1^{-1}$ . The proposition now follows by applying Theorem 3.1 of [H].  $\square$

To ensure the reader that our procedure described as above is practical and makes Hsu's algorithm a perfect choice when  $X$  is given as in subsection 3.1, we provide the following.

**Example 8.2.** Let

$$M_X = \{-\infty, \underset{\bullet}{0}, \underset{\circ}{1}, \underset{\circ}{2}, \underset{\circ}{3}, \infty\} \quad (8.3)$$

be the Hecke-Farey symbols of a subgroup  $X$  of index 11 of  $PSL(2, \mathbb{Z})$  (see Figure 5). Then

$$r_1 = (1, \bar{1})(2, \bar{2})(3, \bar{3})(4, \bar{4}), \quad r_2 = (\bar{1})(1, 5, \bar{2})(2, 6, \bar{3})(3, 7, \bar{4})(4). \quad (8.4)$$

Apply Proposition A.1, the  $TU$ -representation of  $X$  is given as follows.

$$f(T) = (\bar{1}, \bar{2}, \bar{3}, \bar{4}, 4, 7, 3, 6, 2, 5, 1), \quad f(U) = (\bar{1}, 5, \bar{2}, 6, \bar{3}, 7, \bar{4}, 4, 3, 2, 1). \quad (8.5)$$

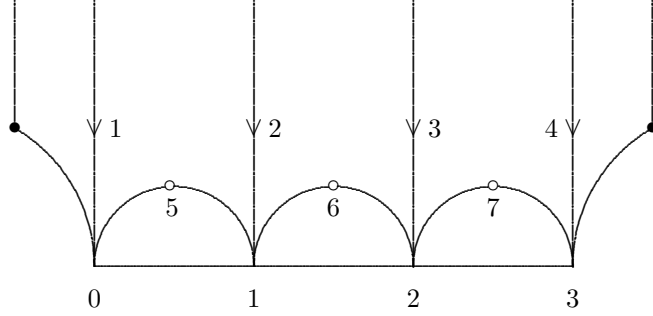


Figure 5

By the algorithm given in Section 3 of [H],  $X$  is non-congruence. By **GAP**,  $\langle f(T), f(U) \rangle \cong A_{11}$  the alternating group on 11 letters. The group  $\cap gXg^{-1}$  was first studied by Magnus [M] as part of his study of non-congruence subgroups of  $PSL(2, \mathbb{Z})$ .

#### APPENDIX A

The following lemma is an application of our study of the Hecke-Farey symbols of the Hecke groups. To the best of our knowledge, this lemma was first proved in [CS] with the help of the celebrated Selberg's Theorem. Our proof is more geometrical and does not make use of Selberg's Theorem.

**Lemma A1.** *Let  $q$  be composite. Then  $G_q$  has infinitely many normal subgroups of finite indices with torsion.*

*Proof.* Recall that  $G_q$  is a free product of  $Z_2 = \langle S \rangle$  and  $Z_q = \langle R_q \rangle$ , where  $R_q = ST^{-1}$ . Suppose that  $q$  is even. Let  $D_{2m} = \langle a, b : a^2 = b^2 = (ab)^m = 1 \rangle$ . Define  $f_m : G_q \rightarrow D_{2m}$  by  $f_m(S) = a$ ,  $f_m(R_q) \rightarrow b$ . It follows easily that the kernel of  $f_m$  is a normal subgroup of index  $2m$  for every  $m$  with torsion. In the case  $q$  is odd, let  $r$  be a prime divisor of  $q$  and let  $X_{2m}$  be a subgroup of  $G_r$  of index  $2rm$  whose additional structure of the Hecke-Farey symbols takes the form  $x_i \underset{a}{\cup} x_{i+1}$  only (a special polygon of  $X_{2m}$  is a union of  $2m$   $r$ -gons with  $2m(r-2)+2$  even lines). It follows that  $X_{2m}$  is torsion free (see subsection 3.1). As a consequence,  $K_{2m} = \cap_{g \in G_r} gX_{2m}g^{-1}$  is normal torsion free of finite index of  $G_r$  and  $G_r/K_{2m}$  is a finite group generated by  $S$  and  $R_r$  of orders 2 and  $r$  respectively modulo  $K_{2m}$ . Define  $f : G_q \rightarrow G_r/K_{2m}$  by  $f(S) = S$ ,  $f(R_q) = R_r$ . It is clear that  $f$  is a surjective homomorphism and that  $G_q$  possesses a normal subgroup  $V_{2m}$  such that  $G_q/V_{2m} \cong G_r/K_{2m}$ . Note that  $(R_q)^r \in V_{2m}$  for all  $m$ . In particular,  $V_{2m}$  is torsion. Consequently,  $G_q$  has infinitely many normal subgroups of finite indices with torsion.  $\square$

**Lemma A2.** *Let  $q \geq 3$  be a prime and  $N$  a normal subgroup of finite index of  $G_q$ . Then  $N$  is either  $G_q$ ,  $G_q^2$  or  $G_q^q$ .*

*Proof.* Suppose that  $N \triangleleft G_q$  has an elliptic element  $g$ . It is well known that  $\langle g \rangle$  is conjugate to either  $\langle S \rangle$  or  $\langle ST^{-1} \rangle$ . Since  $N$  is normal,  $N$  contains all the conjugates of  $\langle g \rangle$ . Hence  $G_q^q = \langle xSx^{-1} : x \in G_q \rangle \subseteq N$  or  $G_q^2 = \langle x(ST^{-1})x^{-1} : x \in G_q \rangle \subseteq N$ . It is easy to see that  $G_q^q$  is a normal subgroup of index  $q$  whose special polygon is the  $q$ -gon

$$\{-\infty \underset{\circ}{\cup} 0 \underset{\circ}{\cup} 1/\lambda_q \underset{\circ}{\cup} \dots \underset{\circ}{\cup} \lambda_q/1 \underset{\circ}{\cup} \infty\} \quad (A1)$$

and that  $G_q^2$  is a normal subgroup of index 2 whose special polygon is the union of two special triangles  $\{-\infty \underset{\bullet}{\cup} 0 \underset{\bullet}{\cup} \infty\}$ . It follows that the only torsion normal subgroups of  $G_q$  are  $G_q$ ,  $G_q^2$  and  $G_q^q$ .  $\square$

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